6-5 Analyzing Graphs of Polynomial Functions

Main Ideas

- Graph polynomial functions and locate their real zeros.
- Find the relative maxima and minima of polynomial functions.

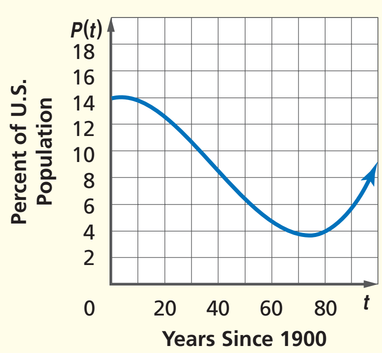
New Vocabulary

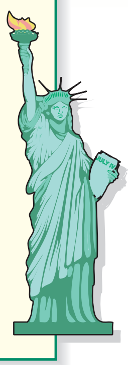
Location Principle
relative maximum
relative minimum

GET READY for the Lesson

The percent of the United States population that was foreign-born since 1900 can be modeled by $P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14$, where $t = 0$ in 1900. Notice that the graph is decreasing from $t = 5$ to $t = 75$ and then it begins to increase. The points at $t = 5$ and $t = 75$ are turning points in the graph.

Foreign-Born Population





Graph Polynomial Functions To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth continuous curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

EXAMPLE Graph a Polynomial Function

Study Tip

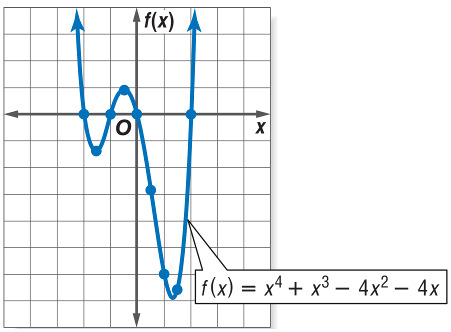
Graphing Polynomial Functions

To graph polynomial functions it will often be necessary to include x-values that are not integers.

1 **Graph $f(x) = x^4 + x^3 - 4x^2 - 4x$ by making a table of values.**

x	$f(x)$
−2.5	≈ 8.4
−2.0	0.0
−1.5	≈ −1.3
−1.0	0.0
−0.5	≈ 0.9

x	$f(x)$
0.0	0.0
0.5	≈ −2.8
1.0	−6.0
1.5	≈ −6.6
2.0	0.0



This is an even-degree polynomial with a positive leading coefficient, so $f(x) \to +\infty$ as $x \to +\infty$, and $f(x) \to +\infty$ as $x \to -\infty$. Notice that the graph intersects the x-axis at four points, indicating there are four real zeros of this function.

CHECK Your Progress

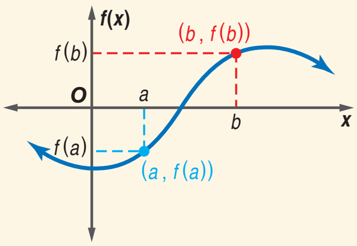
1. Graph $f(x) = x^4 - x^3 - x^2 + x$ by making a table of values.

In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, because it is a continuous function, the graph of a polynomial function will cross the x-axis somewhere between pairs of x-values at which the corresponding $f(x)$-values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x-values. This property for locating zeros is called the **Location Principle**.

KEY CONCEPT — *Location Principle*

Words Suppose $y = f(x)$ represents a polynomial function and a and b are two numbers such that $f(a) < 0$ and $f(b) > 0$. Then the function has at least one real zero between a and b.

Model



EXAMPLE Locate Zeros of a Function

2 **Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 - 5x^2 + 3x + 2$ is located. Then draw the graph.**

Make a table of values. Since $f(x)$ is a third-degree polynomial function, it will have either 1, 2, or 3 real zeros. Look at the values of $f(x)$ to locate the zeros. Then use the points to sketch a graph of the function.

Concepts in Motion
Animation
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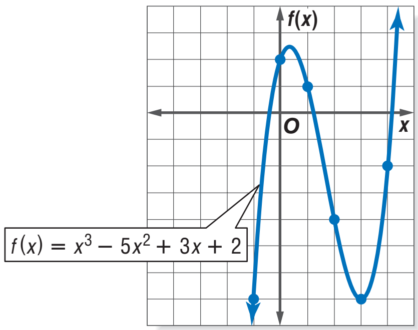
x	$f(x)$	
−2	−32	
−1	−7	change in sign
0	2	
1	1	change in sign
2	−4	
3	−7	
4	−2	change in sign
5	17	



The changes in sign indicate that there are zeros between $x = -1$ and $x = 0$, between $x = 1$ and $x = 2$, and between $x = 4$ and $x = 5$.

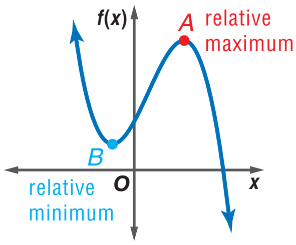
CHECK Your Progress

2. Determine consecutive integer values of x between which each real zero of the function $f(x) = x^3 + 4x^2 - 6x - 7$ is located. Then draw the graph.

Reading Math

Maximum and Minimum The plurals of *maximum* and *minimum* are *maxima* and *minima*.

Maximum and Minimum Points The graph at the right shows the shape of a general third-degree polynomial function.



Point A on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater y-coordinate. Likewise, point B is a **relative minimum** since no other nearby points have a lesser y-coordinate. These points are often referred to as *turning points*. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

EXAMPLE Maximum and Minimum Points

3 **Graph $f(x) = x^3 - 3x^2 + 5$. Estimate the x-coordinates at which the relative maxima and relative minima occur.**

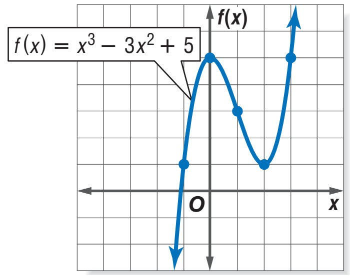
Make a table of values and graph the equation.

x	$f(x)$
−2	−15
−1	1
0	5
1	3
2	1
3	5

zero between $x = -2$ and $x = -1$

← indicates a relative maximum

← indicates a relative minimum



Look at the table of values and the graph.

- The values of $f(x)$ change signs between $x = -2$ and $x = -1$, indicating a zero of the function.
- The value of $f(x)$ at $x = 0$ is greater than the surrounding points, so it is a relative maximum.
- The value of $f(x)$ at $x = 2$ is less than the surrounding points, so it is a relative minimum.

CHECK Your Progress

3. Graph $f(x) = x^3 + 4x^2 - 3$. Estimate the x-coordinates at which the relative maxima and relative minima occur.

The graph of a polynomial function can reveal trends in real-world data.

Real-World Link

Gasoline and diesel fuels are the most familiar transportation fuels in this country, but other energy sources are available, including ethanol, a grain alcohol that can be produced from corn or other crops.

Source: U.S. Environmental Protection Agency

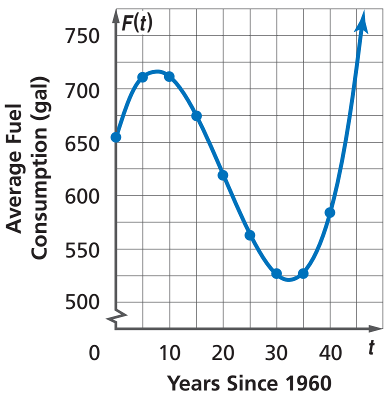
Real-World EXAMPLE Graph a Polynomial Model

4 **ENERGY** **The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation $F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654$, where t is the number of years since 1960.**

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

t	$F(t)$
0	654
5	710.88
10	711.5
15	674.63
20	619
25	563.38
30	526.5
35	527.13
40	584



(continued on the next page)

VCG/Getty Images

b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as t increases, $F(t)$ increases.

c. What trends in fuel consumption does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise. The trend may continue for some years, but it is unlikely that consumption will rise this quickly indefinitely. Fuel supplies will limit consumption.

CHECK Your Progress

4. The price of one share of stock of a company is given by the function $f(x) = 0.001x^4 - 0.03x^3 + 0.15x^2 + 1.01x + 18.96$, where x is the number of months since January 2006. Graph the equation. Describe the turning points of the graph and its end behavior. What trends in the stock price does the graph suggest? Is it reasonable to assume the trend will continue indefinitely?

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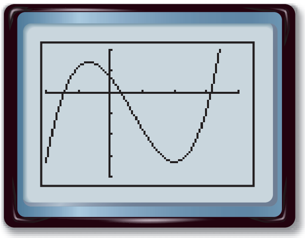
A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.

GRAPHING CALCULATOR LAB

Maximum and Minimum Points

You can use a TI-83/84 Plus to find the coordinates of relative maxima and relative minima. Enter the polynomial function in the Y= list and graph the function. Make sure that all the turning points are visible in the viewing window. Find the coordinates of the minimum and maximum points, respectively.

The graphing calculator screen at the right shows one relative maximum and one relative minimum of the function that is graphed.



KEYSTROKES: *Refer to page 243 to review finding maxima and minima.*

THINK AND DISCUSS

1. Graph $f(x) = x^3 - 3x^2 + 4$. Estimate the x-coordinates of the relative maximum and relative minimum points from the graph.

2. Use the maximum and minimum options from the CALC menu to find the exact coordinates of these points. You will need to use the arrow keys to select points to the left and to the right of the point.

3. Graph $f(x) = \frac{1}{2}x^4 - 4x^3 + 7x^2 - 8$. How many relative maximum and relative minimum points does the graph contain? What are the coordinates?

CHECK Your Understanding

Example 1 (p. 339) **Graph each polynomial function by making a table of values.**

1. $f(x) = x^3 - x^2 - 4x + 4$ **2.** $f(x) = x^4 - 7x^2 + x + 5$

Example 2 (p. 340) **Determine the consecutive integer values of x between which each real zero of each function is located. Then draw the graph.**

3. $f(x) = x^3 - x^2 + 1$ **4.** $f(x) = x^4 - 4x^2 + 2$

Example 3 (p. 341) **Graph each polynomial function. Estimate the x-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.**

5. $f(x) = x^3 + 2x^2 - 3x - 5$ **6.** $f(x) = x^4 - 8x^2 + 10$

Example 4 (pp. 341–342) **CABLE TV** **For Exercises 7–10, use the following information.**

The number of cable TV systems after 1985 can be modeled by the function $C(t) = -43.2t^2 + 1343t + 790$, where t represents the number of years since 1985.

7. Graph this equation for the years 1985 to 2005.

8. Describe the turning points of the graph and its end behavior.

9. What is the domain of the function? Use the graph to estimate the range.

10. What trends in cable TV subscriptions does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Exercises

HOMEWORK HELP

For Exercises	See Examples
11–18	1–3
19–25	4

For Exercises 11–18, complete each of the following.

a. **Graph each function by making a table of values.**

b. **Determine the consecutive integer values of x between which each real zero is located.**

c. **Estimate the x-coordinates at which the relative maxima and relative minima occur.**

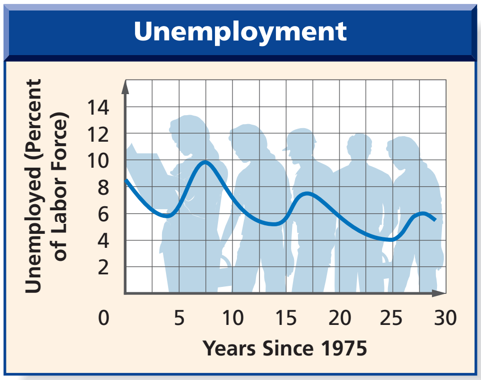
11. $f(x) = -x^3 - 4x^2$ **12.** $f(x) = x^3 - 2x^2 + 6$

13. $f(x) = x^3 - 3x^2 + 2$ **14.** $f(x) = x^3 + 5x^2 - 9$

15. $f(x) = -3x^3 + 20x^2 - 36x + 16$ **16.** $f(x) = x^3 - 4x^2 + 2x - 1$

17. $f(x) = x^4 - 8$ **18.** $f(x) = x^4 - 10x^2 + 9$

EMPLOYMENT **For Exercises 19–22, use the graph that models the unemployment rates from 1975–2004.**



19. In what year was the unemployment rate the highest? the lowest?

20. Describe the turning points and the end behavior of the graph.

21. If this graph was modeled by a polynomial equation, what is the least degree the equation could have?

22. Do you expect the unemployment rate to increase or decrease from 2005 to 2010? Explain your reasoning.

Real-World Link

As children develop, their sleeping needs change. Infants sleep about 16–18 hours a day. Toddlers usually sleep 10–12 hours at night and take one or two daytime naps. School-age children need 9–11 hours of sleep, and teens need at least 9 hours of sleep.

Source: www.kidshealth.org

HEALTH **For Exercises 23–25, use the following information.** During a regular respiratory cycle, the volume of air in liters in human lungs can be described by $V(t) = 0.173t + 0.152t^2 - 0.035t^3$, where t is the time in seconds.

23. Estimate the real zeros of the function by graphing.

24. About how long does a regular respiratory cycle last?

25. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air.

For Exercises 26–31, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of x between which each real zero is located.

c. Estimate the x-coordinates at which the relative maxima and relative minima occur.

26. $f(x) = -x^4 + 5x^2 - 2x - 1$

27. $f(x) = -x^4 + x^3 + 8x^2 - 3$

28. $f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6$

29. $f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5$

30. $f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3$

31. $f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6$

CHILD DEVELOPMENT **For Exercises 32 and 33, use the following information.** The average height (in inches) for boys ages 1 to 20 can be modeled by the equation $B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25$, where x is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation $G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26$.

32. Graph both equations by making a table of values. Use $x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ as the domain. Round values to the nearest inch.

33. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls?

EXTRA PRACTICE
See pages 903, 931.

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Use a graphing calculator to estimate the x-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

34. $f(x) = x^3 + x^2 - 7x - 3$

35. $f(x) = -x^3 + 6x^2 - 6x - 5$

36. $f(x) = -x^4 + 3x^2 - 8$

37. $f(x) = 3x^4 - 7x^3 + 4x - 5$

H.O.T. Problems

38. **OPEN ENDED** Sketch a graph of a function that has one relative maximum point and two relative minimum points.

CHALLENGE **For Exercises 39–41, sketch a graph of each polynomial.**

39. even-degree polynomial function with one relative maximum and two relative minima

40. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative

41. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

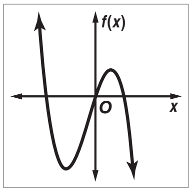
42. **REASONING** Explain the Location Principle and how to use it.

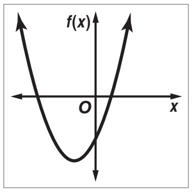
43. ***Writing in Math*** Use the information about foreign-born population on page 339 to explain how graphs of polynomial functions can be used to show trends in data. Include a description of the types of data that are best modeled by polynomial functions and an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.

Michael Newman/PhotoEdit

STANDARDIZED TEST PRACTICE

44. ACT/SAT Which of the following could be the graph of $f(x) = x^3 + x^2 - 3x$?

A


C


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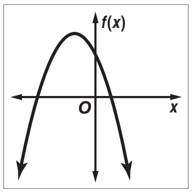

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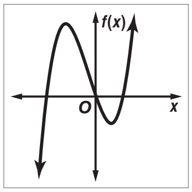


45. REVIEW Mandy went shopping. She spent two-fifths of her money in the first store. She spent three-fifths of what she had left in the next store. In the last store she visited, she spent three-fourths of the money she had left. When she finished shopping, Mandy had \$6. How much money in dollars did Mandy have when she started shopping?

F \$16 **H** \$100

G \$56 **J** \$106

Spiral Review

If $p(x) = 2x^2 - 5x + 4$ and $r(x) = 3x^3 - x^2 - 2$, find each value. (Lesson 6-4)

46. $r(2a)$ **47.** $5p(c)$ **48.** $p(2a^2)$

49. $r(x - 1)$ **50.** $p(x^2 + 4)$ **51.** $2[p(x^2 + 1)] - 3r(x - 1)$

Simplify. (Lesson 6-3)

52. $(4x^3 - 7x^2 + 3x - 2) \div (x - 2)$ **53.** $\dfrac{x^4 + 4x^3 - 4x^2 + 5x}{x - 5}$

Simplify. (Lesson 6-2)

54. $(3x^2 - 2xy + y^2) + (x^2 + 5xy - 4y^2)$ **55.** $(2x + 4)(7x - 1)$

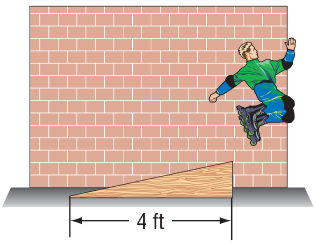
Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

56. $\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}$ **57.** $\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

58. $3j + 2k = 8$
$j - 7k = 18$

59. $5y + 2z = 11$
$10y - 4z = -2$

60. SPORTS Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of $\frac{1}{4}$, how tall should they make the ramp? (Lesson 2-3)



GET READY for the Next Lesson

PREREQUISITE SKILL **Find the greatest common factor of each set of numbers.**

61. 18, 27 **62.** 24, 84 **63.** 16, 28

64. 12, 27, 48 **65.** 12, 30, 54 **66.** 15, 30, 65